Solution for HW8

23-11-2016

§59) 2) a) Since $f(z)=e^z$, $f^{(n)}(z)=e^z$ and $f^{(n)}(1)=e$

So the Talyor series at $z=1$ is given by

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

b) $$e^z = e^{z-1}\cdot e = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}.$$

8) $$\cos z = -\sin\left(z-\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty}\frac{(-1)^n\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}.$$

13) $$\frac{1}{4z-z^2} = \frac{1}{4z}+\frac{1}{4(4-z)}$$

$$= \frac{1}{4z}+\frac{1}{4^2\left(1-\frac{z}{4}\right)}$$

$$= \frac{1}{4z}+\frac{1}{4^2}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$$

$$= \frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

§62) 1) $$f(z)=z^2\sin\left(\frac{1}{z^2}\right)=z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z^2}\right)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}$$

$$\Rightarrow f(z)= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}.$$

3) $$f(z)=\frac{1}{1+z}=\frac{1}{z}\cdot\frac{1}{1+\left(\frac{1}{z}\right)}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

$$\Rightarrow f(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}.$$

4) On $0<|z|<1$,

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{1}{z^2}\sum_{n=0}^{\infty}z^n=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty}z^n.$$

On $1<|z|<+\infty$, we have $\frac{1}{|z|}<1$.

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{-1}{z^3}\cdot\frac{1}{1-\frac{1}{z}}=\frac{-1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=3}^{\infty}\frac{1}{z^n}$$

6) On $0<|z-1|<2$, we have $\frac{|z-1|}{2}<1$.

$$\frac{z}{(z-1)(z-3)}=\frac{(z-1)+1}{(z-1)(z-3)}$$

$$=\frac{1}{z-3}+\frac{1}{(z-1)(z-3)}$$

$$=\frac{-1}{2-(z-1)}-\frac{1}{(z-1)(2-(z-1))}$$

$$=\frac{-1}{2}\left(\frac{1}{1-\frac{z-1}{2}}+\frac{1}{(z-1)(1-\frac{z-1}{2})}\right)$$

$$=\frac{-1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n+\frac{\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n}{(z-1)}\right)$$

$$=\frac{-1}{2}\left(\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}+\sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{2^n}\right)$$

$$=\frac{-1}{2}\left(\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{2^{n-1}}\right)$$

$$=\frac{-1}{2}\left(\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}+\frac{1}{z-1}\right)$$

$$=-\frac{3}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}}-\frac{1}{2(z-1)}.$$

$$=-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}-\frac{1}{2(z-1)}$$

8) a) On $|a|<|z|<+\infty$, we have $\frac{|a|}{|z|}<1$.

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-\frac{a}{z}}$$

$$= \frac{a}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{a}{z}\right)^n$$

b) Put $z = e^{i\theta}$. By a),

$$L.H.S. = \frac{a}{e^{i\theta}-a} = \frac{a}{(\cos\theta - a) + i\sin\theta} = \frac{a[(\cos\theta - a) - i\sin\theta]}{(\cos\theta - a)^2 + \sin^2\theta}$$

$$\Rightarrow L.H.S. = \frac{a\cos\theta - a^2}{1-2a\cos\theta + a^2} - i\,\frac{a\sin\theta}{1-2a\cos\theta+a^2}$$

$$R.H.S. = \sum_{n=1}^{\infty} \frac{a^n}{(e^{i\theta})^n} = \sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos n\theta - i \sum_{n=1}^{\infty} a^n \sin n\theta$$

Comparing the real and imaginary parts on both sides, we have

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1-2a\cos\theta + a^2} \\ \displaystyle\sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1-2a\cos\theta + a^2} \end{cases}$$

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